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# AXISYMMETRIC QUASI-STATIC THERMAL STRESSES IN AN INFINITE SLAB

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**Abstract**—When a slab, initially undisturbed, is subjected to a step-functional and axi-symmetrical heat-flow from one face and another face is insulated, the thermoelastic problem is analyzed by the parametric uses of a thermoelastic potential and a stress function. Numerical calculations of the temperature and the thermal stress distributions are carried out for the case of a circular region being subjected to a uniform heat-flow and it is considered how the ratio of the radius of the heated region to the plate thickness affects on their distributions.

Moreover, the approximate solution based upon the theory of plates is introduced in order to compare the rigorous one. It can be clarified that for a small time interval the solution based upon the theory of plates has a good approximation when the plate can be regarded as a thin plate in contrast with the dimension of any heated region.

## NOTATION

| | |
|---|---|
| $\sigma_r$ | radial stress |
| $\sigma_\theta$ | circumferential stress |
| $\tau_{rz}$ | shear stress |
| $\sigma^p$ | stress by the theory of plates |
| $h$ | plate thickness |
| $\kappa$ | thermal diffusivity |
| $T$ | temperature |
| $t$ | time |
| $K$ | thermal conductivity |
| $Q(r)$, $Q_0$ | heat-flux density |
| $a$ | radius of a heated region |
| $(r, z)$ | cylindrical co-ordinates |
| $(u, w)$ | components of the displacement vector in $r$- and $z$-directions |
| $(\rho, \zeta)$ | dimensionless co-ordinates; $\rho = r/h$, $\zeta = z/h$ |
| $\tau$ | dimensionless time; $\tau = \kappa t/h^2$ |
| $T/T^*$ | dimensionless temperature; $T^* = Q_0 h/k$ |
| $\rho_0$ | dimensionless parameter; $\rho_0 = a/h$ |
| $e_0$ | cubical dilatation; $e_0 = \dfrac{\partial u}{\partial r}+\dfrac{u}{r}+\dfrac{\partial w}{\partial z}$ |
| $\nu$ | Poisson's ratio |
| $\alpha$ | coefficient of linear thermal expansion |
| $E$ | modulus of elasticity |
| $\Omega$ | thermoelastic potential |
| $\psi$ | stress function |
| $\beta$ | parameter; $\beta^2 = \xi^2 + p/\kappa$ |
| $\nabla^2$ | Laplacian operator; $\nabla^2 \equiv \dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{\partial^2}{\partial z^2}$ |
| $J_\eta(x)$ | Bessel function of the first kind in order $\eta$ |

## 1. INTRODUCTION

THE surfaces of some machine elements will happen to be partially heated in casting and welding works and so on. Thereafter, in order to suggest the design of those elements, it will be very significant to clarify the temperature field and the thermal stress distributions in a slab whose face is partially heated. Many investigations on such problems have been reported. For example, Heisler [1] has introduced the solution of the thermoelastic problem to a plate with one face exposed to heat input. This was followed by the work of Muki [2] who has derived the thermal stress solution in a thick plate with steady distribution of temperature. Although the boundary conditions in both investigations have been formulated by the surface temperature, the surface temperature of a body heated by fused metal will vary as a function of time and space, which cannot be easily known. Therefore, studying such a problem by defining the surface heat-flow, one may obtain the practical results through the simple formulation of boundary condition. Koizumi [3] has analyzed the quasi-static thermal stresses in a semi-infinite body whose surface is subjected to an axi-symmetrical heat-flow. But the investigation for a thick plate has been little given in spite of the availability of its results.

In the present paper, when an infinite slab, initially at zero temperature throughout, is subjected to a step-functional and axi-symmetrical heat-flow from one face and the other face is insulated, the thermoelastic problem is analyzed by the parametric uses of a thermoelastic potential and a stress function. The Fourier's heat conduction equation and the equilibrium equations without the inertia terms are employed, and also the Laplace transformation is introduced for the sake of the analytical simplicity. Numerical calculations of the temperature and the thermal stress distributions are carried out only for the case of a circular region being heated uniformly and it is considered how the ratio of the radius of a heated region to the plate thickness affects on their distributions.

## 2. TEMPERATURE

Strictly speaking, the heat conduction problem in a stress field must be analyzed on the basis of the coupled heat conduction equation. However, it is well recognized that the Fourier's heat conduction equation in the absence of the coupled term may be taken instead of it in the thermoelastic problem. Besides the above recognition, in the present study, we suppose that the physical properties of substances are independent of temperature. Now we make the $z$-axis coincide with the symmetrical one. Then the Fourier's heat conduction equation is given as follows:

$$\nabla^2 T = \frac{1}{\kappa}\frac{\partial T}{\partial t}. \tag{1}$$

Since no thermal stress occurs in the uniform temperature field, we suppose that the initial temperature in the body is maintained to be uniform, taking it as the equilibrium temperature. Hence, in this case, the initial condition will be directly written.

$$(T)_{t=0} = 0. \tag{2}$$

Now, let's call the face $z = 0$ "the surface I" and the face $z = h$ "the surface II". Thereafter we assume that the surface II is insulated and the surface I is heated by an axi-symmetrical heat-flow of the constant rate $Q(r)$/unit area per unit time, which varies as the step-function of time. Then the boundary conditions are written as follows:

$$\left(K\frac{\partial T}{\partial z}\right)_{z=0} = -Q(r) \qquad (t > 0) \tag{3}$$

$$\left(K\frac{\partial T}{\partial z}\right)_{z=h} = 0 \qquad (t > 0). \tag{4}$$

Additionally,

$$(T)_{r\to\infty} = 0 \qquad (t > 0). \tag{5}$$

Now we introduce the Laplace transformation to derive the temperature solution. Let the Laplace transform of $T$ be represented by $\bar{T} = \int_0^\infty T\,\mathrm{e}^{-pt}\,\mathrm{d}t$. In consideration of the initial condition (2) the transform of equation (1) can be written as follows:

$$(\nabla^2 - p/\kappa)\bar{T} = 0. \tag{6}$$

The solution of equation (6) satisfying the boundary conditions (3)–(5) can be easily found as

$$\bar{T} = \int_0^\infty \xi\hat{Q}(\xi)J_0(r\xi)\frac{\cosh(h-z)\beta}{pK\beta\sinh(h\beta)}\,\mathrm{d}\beta. \tag{7}$$

Where

$$\hat{Q}(\xi) = \int_0^\infty rJ_0(r\xi)Q(r)\,\mathrm{d}r.$$

Performing the inverse integral of equation (7), we derive the temperature solution.

$$T = \int_0^\infty \frac{\hat{Q}(\xi)}{K}J_0(\xi r)\left[\frac{\cosh(h-z)\xi}{\sinh(h\xi)} - \frac{\mathrm{e}^{-\kappa\xi^2 t}}{h\xi}\left\{1 + 2h^2\xi^2\sum_{n=1}^{\infty}\frac{\cos(n\pi z/h)}{(n^2\pi^2+\xi^2h^2)}\mathrm{e}^{-\kappa n^2\pi^2 t/h^2}\right\}\right]\mathrm{d}\xi. \tag{8}$$

## 3. STRESSES

In the present study, we do not consider the thermal shock phenomenon and moreover we suppose that the moduli of elasticity are constant. Thus we analyze the thermal stresses induced in the slab due to the temperature distribution given by equation (8).

The basic equations of thermoelasticity in the axisymmetric field are represented as follows:

$$\begin{aligned}\nabla^2 u + \frac{1}{1-2\nu}\frac{\partial e_0}{\partial r} - \frac{u}{r^2} &= 2\frac{1+\nu}{1-2\nu}\cdot\alpha\frac{\partial T}{\partial r}\\ \nabla^2 w + \frac{1}{1-2\nu}\frac{\partial e_0}{\partial z} &= 2\frac{1+\nu}{1-2\nu}\cdot\alpha\frac{\partial T}{\partial z}.\end{aligned} \tag{9}$$

Now, let's make use of $\Omega$ and $\psi$, which are defined as follows:

$$\begin{aligned}\frac{1-\nu}{1+\nu}\frac{u}{\alpha} &= \frac{\partial\Omega}{\partial r} + z\frac{\partial\psi}{\partial r}\\ \frac{1-\nu}{1+\nu}\frac{w}{\alpha} &= \frac{\partial\Omega}{\partial z} + z\frac{\partial\psi}{\partial z} - (3-4\nu)\psi\end{aligned} \tag{10}$$

Where

$$\nabla^2 \Omega = T \tag{11}$$

$$\nabla^2 \psi = 0. \tag{12}$$

When the body is free from loading, the boundary conditions on equilibrium are shown by the equations

$$\begin{aligned} (\bar{\sigma}_z)_{z=0} &= 0, \qquad (\bar{\tau}_{rz})_{z=0} = 0 \\ (\bar{\sigma}_z)_{z=h} &= 0, \qquad (\bar{\tau}_{rz})_{z=h} = 0. \end{aligned} \tag{13}$$

Additionally

$$(\bar{\sigma}, \bar{\tau})_{r\to\infty} = 0. \tag{14}$$

Substituting the solutions of equations (11) and (12) into the Laplace transform of equation (10), we can derive the solution of the displacements in the Laplace transforms. Moreover, using the Duhamel–Neumann law, we get the solution of the stresses. Substituting the results into equations (13) and (14), we can determine the unknown constants in the expression of the solutions. Performing the inverse integrals of those results, we obtain the desired solutions of the displacements and the stresses.

$$\begin{aligned} \frac{1-\nu}{1+\nu}\frac{u}{\alpha} &= \int_0^\infty \frac{\hat{Q}(\xi)}{K} J_1(\xi r)\left[\frac{(1-\nu)\cosh(h-z)\xi}{\xi\sinh(h\xi)} - 2h^3\xi^2 \mathrm{e}^{-\kappa\xi^2 t}\left\{\frac{f(0,z)}{2} + \sum_{n=1}^\infty f(n,z)\right\}\right]\mathrm{d}\xi \\ \frac{1-\nu}{1+\nu}\frac{w}{\alpha} &= \int_0^\infty \frac{\hat{Q}(\xi)}{K} J_0(\xi r)\left[\frac{(\nu-1)\sinh(h-z)\xi}{\xi\sinh(h\xi)} - 2h^3\xi^2 \mathrm{e}^{-\kappa\xi^2 t}\left\{\frac{x(0,z)}{2} + \sum_{n=1}^\infty x(n,z)\right\}\right]\mathrm{d}\xi \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{1-\nu}{E\alpha}\sigma_r &= \int_0^\infty \frac{\hat{Q}(\xi)}{K}\left[\frac{(\nu-1)\cosh(h-z)\xi}{\sinh(h\xi)}\frac{J_1(\xi r)}{\xi r}\right. \\ &\quad \left. + 2h^3\xi^3 \mathrm{e}^{-\kappa\xi^2 t}\left\{\frac{J_1(\xi r)}{\xi r}\left(\frac{f(0,z)}{2} + \sum_{n=1}^\infty f(n,z)\right) + J_0(\xi r)\left(\frac{g(0,z)}{2} + \sum_{n=1}^\infty g(n,z)\right)\right\}\right]\mathrm{d}\xi \\ \frac{1-\nu}{E\alpha}\sigma_\theta &= \int_0^\infty \frac{\hat{Q}(\xi)}{K}\left[\frac{(\nu-1)\cosh(h-z)\xi}{\sinh(h\xi)}\left\{J_0(\xi r) - \frac{J_1(\xi r)}{\xi r}\right\}\right. \\ &\quad + 2h^3\xi^3 \mathrm{e}^{-\kappa\xi^2 t}\left\{J_0(\xi r)\left(\frac{g(0,z)}{2} + \sum_{n=1}^\infty g(n,z)\right)\right. \\ &\quad \left.\left. + \left(J_0(\xi r) - \frac{J_1(\xi r)}{\xi r}\right)\left(\frac{f(0,z)}{2} + \sum_{n=1}^\infty f(n,z)\right)\right\}\right]\mathrm{d}\xi \\ \frac{1-\nu}{E\alpha}\sigma_z &= \int_0^\infty \frac{2\hat{Q}(\xi)}{K} J_0(\xi r) h^3\xi^3 \mathrm{e}^{-\kappa\xi^2 t}\left\{\frac{v(0,z)}{2} + \sum_{n=1}^\infty v(n,z)\right\}\mathrm{d}\xi \\ \frac{1-\nu}{E\alpha}\tau_{rz} &= \int_0^\infty \frac{2\hat{Q}(\xi)}{K} J_1(\xi r) h^3\xi^3 \mathrm{e}^{-\kappa\xi^2 t}\left\{\frac{y(0,z)}{2} + \sum_{n=1}^\infty y(n,z)\right\}\mathrm{d}\xi \end{aligned} \tag{16}$$

$$\omega(n,z) = \frac{\cos(n\pi z/h)}{(n^2\pi^2+\xi^2h^2)^2}\,\mathrm{e}^{-\kappa n^2\pi^2 t/h^2}$$

$$U(z) = \frac{hz\xi^2\sinh(h-z)\xi+(h-z)\xi\sinh(h\xi)\sinh(z\xi)}{\sinh^2(h\xi)-h^2)^2} \tag{17}$$

$$V(z) = \frac{2\{h\xi\sinh(h-z)\xi-\sinh(h\xi)\sinh(z\xi)\}}{\sinh^2(h\xi)-h^2\xi^2}$$

$$f(n,z) = \omega(n,z)+\{\nu\xi V(z)-U'(z)\}\omega(n,h)/\xi+\{\nu\xi V(h-z)-U'(h-z)\}\omega(n,0)/\xi$$

$$g(n,z) = n^2\pi^2\omega(n,z)/h^2\xi^2+\{U'(z)\omega(n,h)+U'(h-z)\omega(n,0)\}/\xi$$

$$v(n,z) = \omega(n,z)-\{U'(z)\omega(n,h)+U'(h-z)\omega(n,0)\}/\xi \tag{18}$$

$$y(n,z) = U(z)\omega(n,z)-U(h-z)\omega(n,0)-\omega'(n,z)/\xi$$

$$x(n,z) = y(n,z)+(\nu\xi-1)\{V'(z)\omega(n,h)+V'(h-z)\omega(n,0)\}/\xi \quad (\ )' = \mathrm{d}(\ )/\mathrm{d}z.$$

## 4. NUMERICAL CALCULATIONS

In order to clarify the temperature and the stress distributions, numerical calculations are carried out for the case of a circular region being heated by the uniform heat-flow of the constant rate $Q_0$/unit area per unit time.

We put $\nu = 0{\cdot}3$ in calculations. The definite integrals in equations (8) and (16) have been evaluated by making use of Simpson's first rule. Then, we devise to keep the errors in the numerical integrations less than 1 per cent. For example, in the case of $\rho = 0$, $\zeta = 0$, $\rho_0 = 1$ and $\tau = 0{\cdot}01$, the numerical integration has been computed for $h\xi = 0(0{\cdot}1)50$, and the infinite series has been evaluated for $n = 1, 2, 3\ldots10$. Then, the errors become less than 1 per cent.

First, the thermal stress distribution $\sigma_r$ and $\sigma_\theta$ induced on the surface I are calculated for four different values of $\rho_0 = 0{\cdot}1$, $0{\cdot}4$, $1{\cdot}0$ and $2{\cdot}0$, which are shown in Figs. 1–4. In all

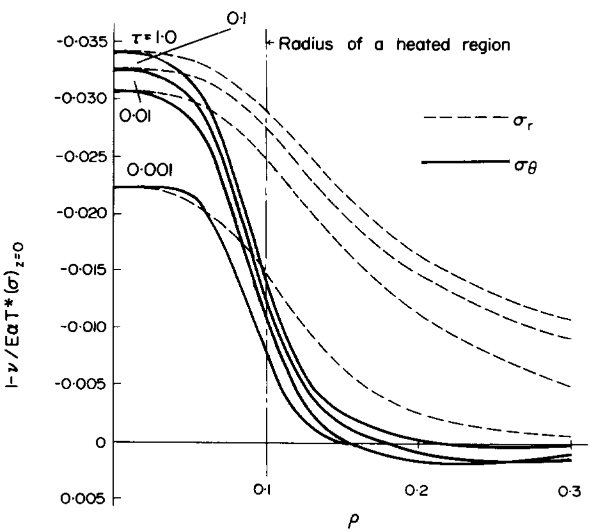


FIG. 1. Stress distributions on the heated surface I ($\rho_0 = 0{\cdot}1$).

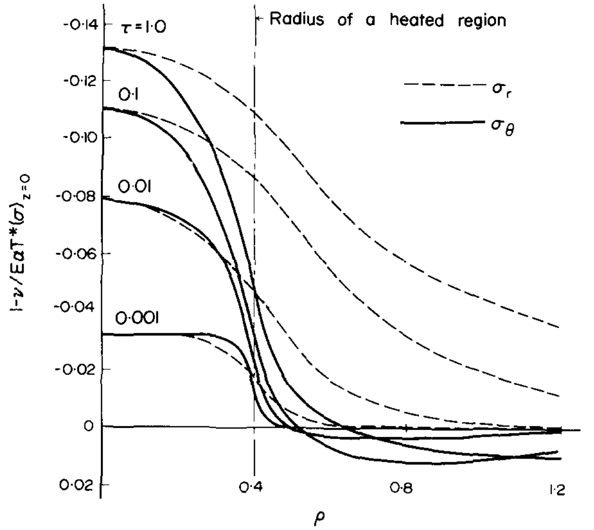


FIG. 2. Stress distributions on the heated surface I ($\rho_0 = 0{\cdot}4$).

the cases, the stresses increase with the progress of time. For a small value of $\rho_0$, the stress distributions in the heated region are uniform only in a small time interval, namely, the Fourier's number $\tau$ is small. To the contrary, in the case of a large value of $\rho_0$, the stresses in the heated region continue to keep the uniform distributions after a long time elapsed. Moreover, $\sigma_r$ at the origin in co-ordinates is identically equal to $\sigma_\theta$ at the same point, and its stress becomes the maximum of the stresses induced on the surface. In a region of

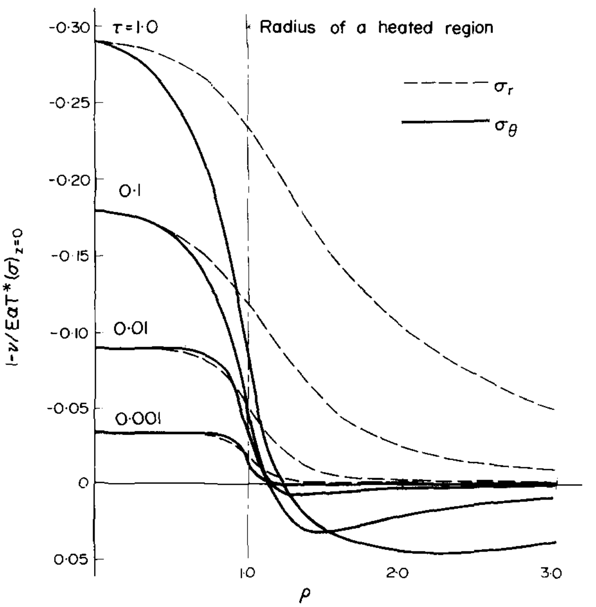


FIG. 3. Stress distributions on the heated surface I ($\rho_0 = 1{\cdot}0$).

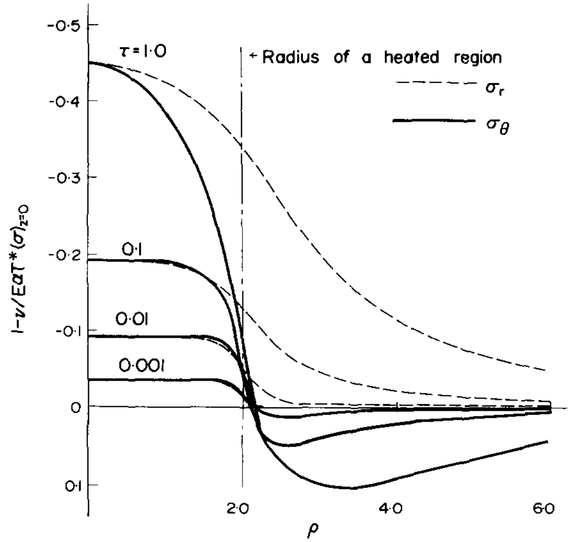


FIG. 4. Stress distributions on the heated surface I ($\rho_0 = 2{\cdot}0$).

$r > a$, the tensile circumferential stresses occur and their values increase as $\rho_0$ becomes large.

Next we have calculated the thermal stress distributions: $(\sigma_r)_{z=h}$ and $(\sigma_\theta)_{z=h}$ on the surface II in two cases of $\rho_0 = 0{\cdot}4$ and $1{\cdot}0$, showing the results in Figs. 5 and 6. The stress distributions slowly decrease and the values are smaller than those on the surface I.

Figures 7 and 8 show the thermal stresses: $(\sigma_r)_{r=0,z=0} = (\sigma_\theta)_{r=0,z=0}$ and $(\sigma_r)_{r=0,z=h} = (\sigma_\theta)_{r=0,z=h}$ at the points where the symmetrical axis intersects the surfaces I and II.

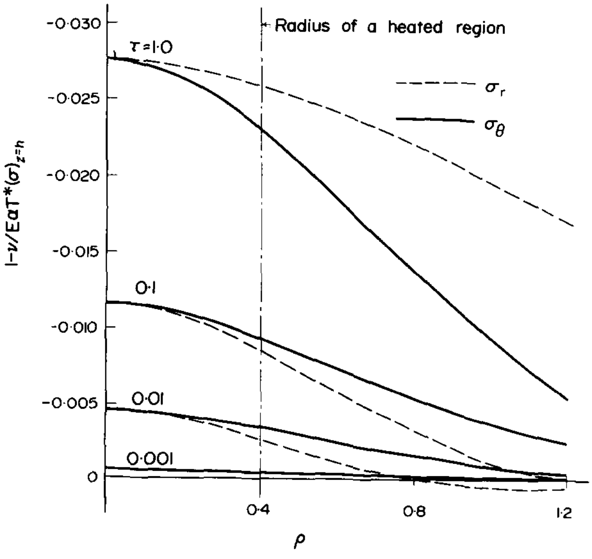


FIG. 5. Stress distributions on the insulated surface II ($\rho_0 = 0{\cdot}4$).

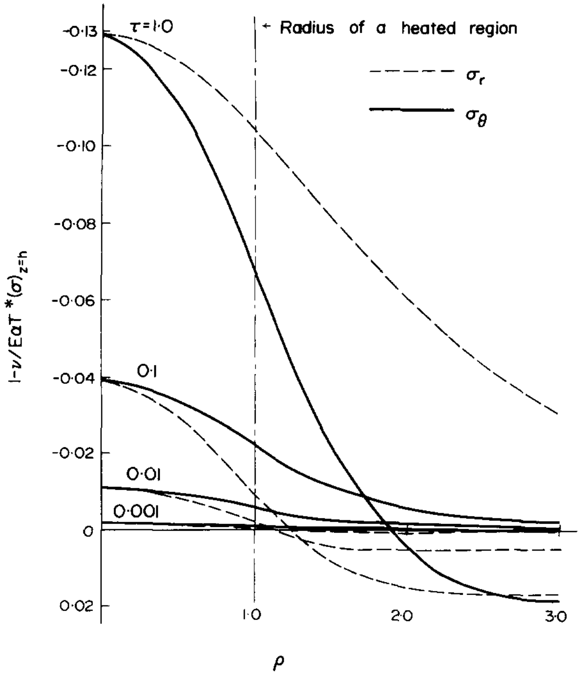


FIG. 6. Stress distributions on the insulated surface II ($\rho_0 = 1{\cdot}0$).

As a value of $\rho_0$ becomes small, the increasing rate of the stresses becomes small rapidly.

Another interest is the temperature distributions $(T)_{z=0}$ on the surface I which calculated for each value of $\rho_0 = 0{\cdot}1, 0{\cdot}4, 1{\cdot}0$ and $2{\cdot}0$, and their results are shown in Figs. 9–12, respectively. The temperature distributions at a small time interval are nearly uniform in the heated region, and those values are approximately equal to zero in the remaining one.

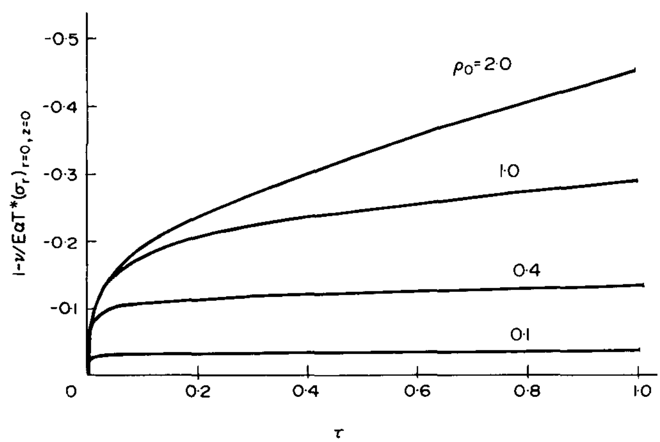


FIG. 7. Variations of the thermal stresses at the origin of the co-ordinates on the heated surface.

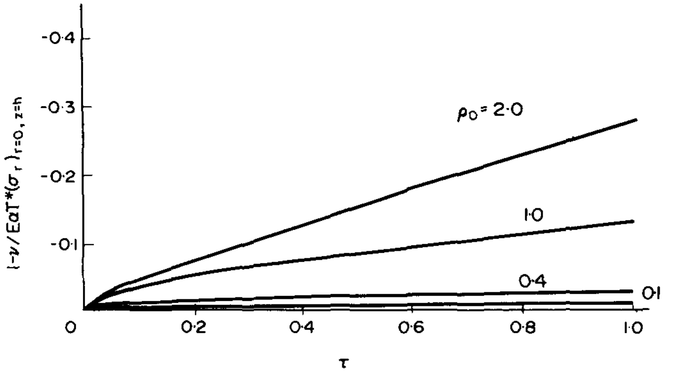


FIG. 8. Variations of the thermal stresses at the point where the symmetrical axis intersect the insulated surface II.

As the time goes by, the temperature in the heated region gradually increases, especially the temperature at the origin being the most predominant. At the same time the temperature in the remaining region increases gradually too. This indicates that plenty of heat begins to flow to the radial direction.

Lastly, we introduce not only the rigorous solution but also the approximate one by the theory of plates [4]. The thermal stresses induced by the temperature field (8) are given as follows:

$$\begin{aligned} \frac{1-\nu}{E\alpha}\sigma_r^p &= \frac{1-\nu}{K}\int_0^\infty \frac{\hat{Q}(\xi)}{\xi h}\left\{\frac{J_1(\xi r)}{\xi r}-\frac{J_0(\xi r)}{1-\nu}\right\}F(\xi, z, t)\,\mathrm{d}\xi - T \\ \frac{1-\nu}{E\alpha}\sigma_\theta^p &= \frac{1-\nu}{K}\int_0^\infty \frac{\hat{Q}(\xi)}{\xi h}\left\{\frac{\nu}{\nu-1}J_0(\xi r)-\frac{J_1(\xi r)}{\xi r}\right\}F(\xi, z, t)\,\mathrm{d}\xi - T \end{aligned} \tag{19}$$

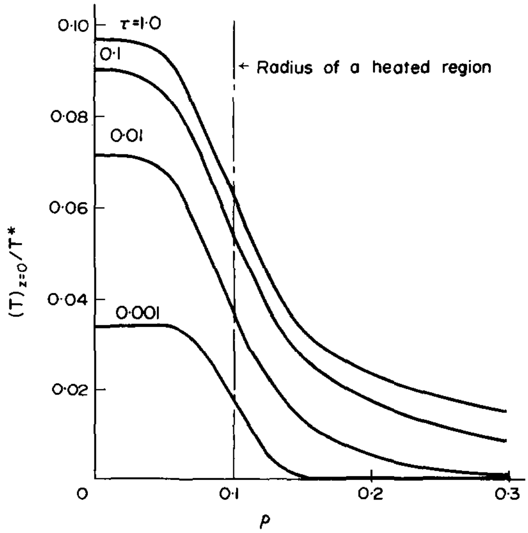


FIG. 9. Temperature distributions on the heated surface I ($\rho_0 = 0{\cdot}1$).

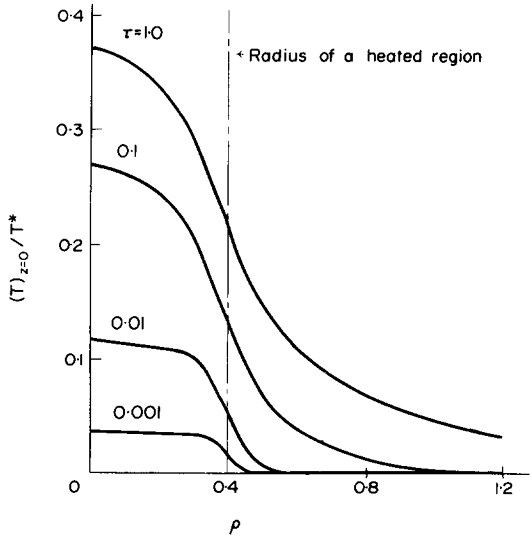


FIG. 10. Temperature distributions on the heated surface I ($\rho_0 = 0{\cdot}4$).

where

$$F(\xi, z, t) = e^{-\kappa\xi^2 t} - 1 - 12\xi\left(\frac{h}{2} - z\right)\left\{\frac{1}{2\xi h} + \frac{1-\cosh(\xi h)}{\xi^2 h^2 \sinh(\xi h)} - \frac{4\xi h}{\pi^2} e^{-\kappa\xi^2 t} \sum_{n=1,3,5\ldots}^{\infty} \frac{e^{-\kappa n^2\pi^2 t/h^2}}{n^2(n^2\pi^2 + \xi^2 h^2)}\right\}. \tag{20}$$

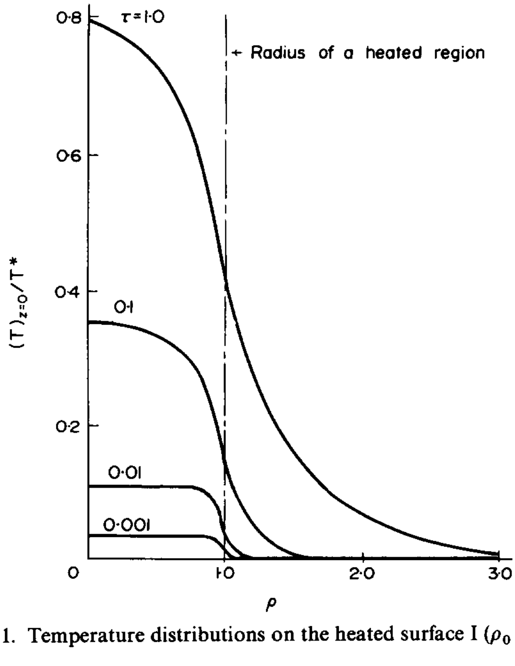


FIG. 11. Temperature distributions on the heated surface I ($\rho_0 = 1{\cdot}0$).

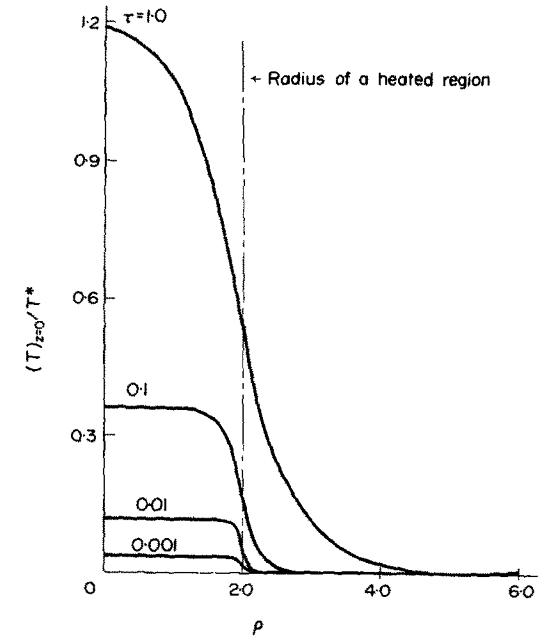


FIG. 12. Temperature distributions on the heated surface I ($\rho_0 = 2{\cdot}0$).

The stresses $(\sigma_r^p)_{r=0,z=0} = (\sigma_\theta^p)_{r=0,z=0}$ and $(\sigma_r^p)_{r=0,z=h} = (\sigma_\theta^p)_{r=0,z=h}$ are calculated by making use of equation (19). The numerical results are shown in Figs. 13 and 14, in which the rigorous solutions are indicated by the solid curve in the same figure. It can be seen that the approximate solution tends to close to the rigorous one. Especially, when $\tau$ is small

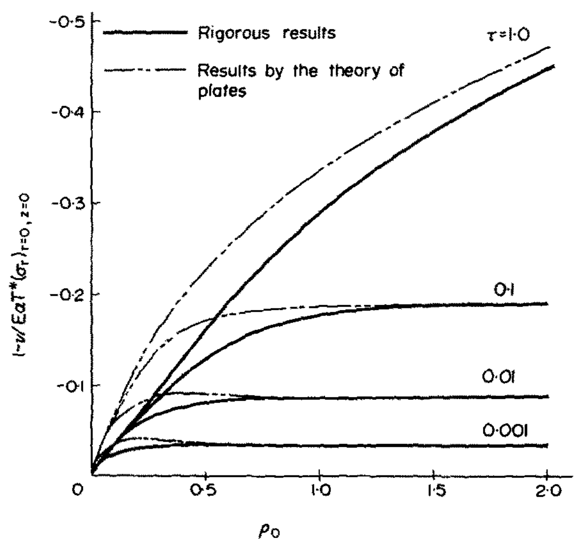


FIG. 13. The comparison between the rigorous and the approximate results at the origin of the co-ordinates on the heated surface I.

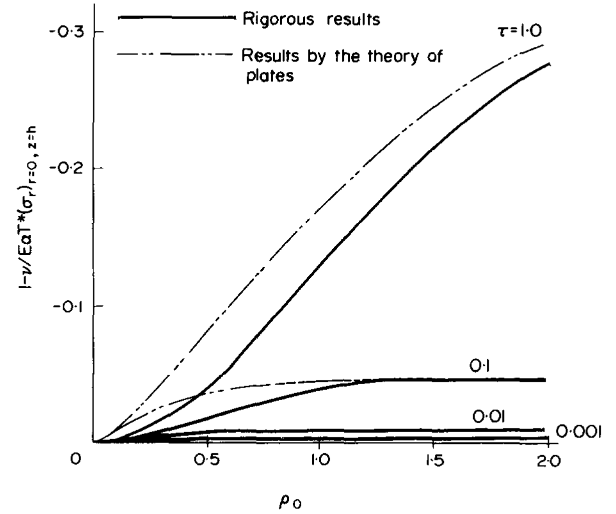


FIG. 14. The comparison between the rigorous and the approximate results at the point where the symmetrical axis intersects the insulated surface II.

and $\rho_0$ is large, there is a good agreement between both the results. Additionally, we have tried to obtain the thermal stress distributions on the symmetrical axis only in the case of $\rho_0 = 1{\cdot}0$, and the results are shown in Fig. 15. From this figure it can be said that for a small time interval both have a good agreement but they deviate from each other as the time goes by.

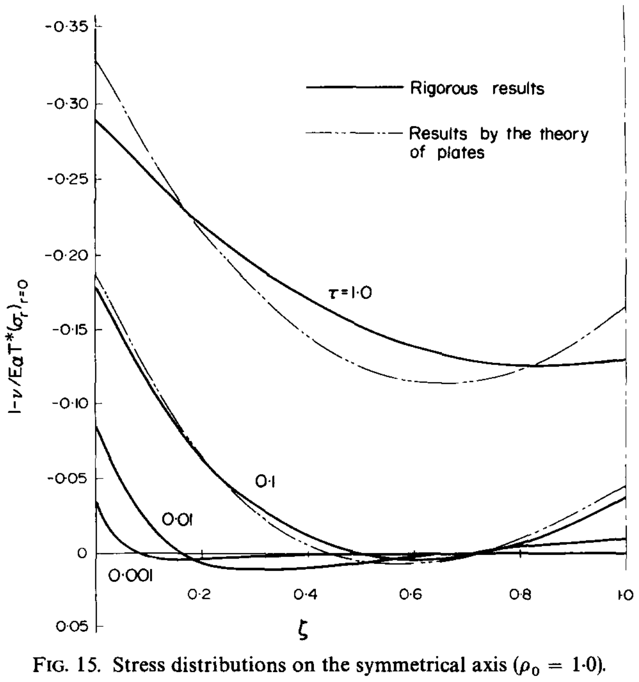


FIG. 15. Stress distributions on the symmetrical axis ($\rho_0 = 1{\cdot}0$).

## 5. CONCLUSIONS

In the present paper the thermoelastic problem of an infinite, thick plate with the face exposed to the partial and axisymmetrical heat-flow has been analyzed and numerical calculations of the temperature and the stress distributions have been carried out under the simple boundary conditions. We also introduce the thermal stresses by the theory of plates. Thus we can conclude the following expressions from those results.

1. By comparing the results of the rigorous solution with the approximate solution, we can clarify that the solution by the theory of plates have a good approximation in case of a thin plate, in contrast with the dimension of any heated region and a small time interval.

2. The thermoelastic problem in the present study can be introduced by the parametric use of a thermoelastic potential and a stress function.

3. Under the thermal boundary conditions in the present study, the thermal stresses induced on the heated surface is larger than those on the insulated surface. Moreover, the maximum compressive thermal stress occurs in the center of the heated surface and the tensile circumferential stresses appear at a far distance from the origin.

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**Абстракт**—Для случая когда плита, начально невозмущенная, является подверженная действию потока тепла, в виде скачкообразной функции и осесимметрическому, с одной стороны, и другая сторона изолированная, тогда задача исследуется путем параметрического использования термоупругого потенциала и функции напряжений. Приводятся численные расчеты распределений температуры и термических напряжений, для случая круглого района, полверженного действию однородного потока тепла. Обсуждается влияние отношения радиуса нагретого района к толщине плиты на распределение напряжений.

Далее, определяется приближенное решение на основе теории пластинок, с целью сравнения со строгим решением. Можно выяснить что для малого интервала времени решение на основе теории пластинок имеет хорошое приближение, когда плита рассматривается как тонкая пластинка по сравнению с размером любого нагретого района.